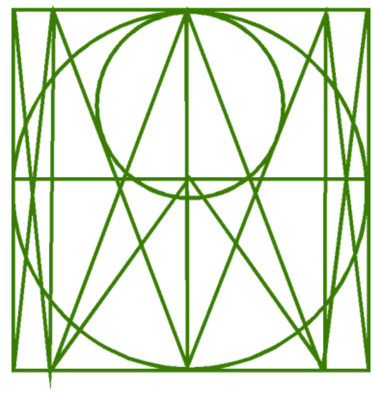
1

RATIONAL FUNCTIONS

Remember what we call a number like $\frac{2}{7}$? This is called a *rational* number because it is the *ratio* of two integers. In a like manner, a rational function is the ratio of two special functions called polynomials. Since a rational function is essentially a fraction, we will have to avoid dividing by zero, which means the domain may not be all real numbers.



❑ POLYNOMIAL FUNCTIONS

Each of the following is a polynomial function:

$y = 7$ (a *linear* function — it's a horizontal line)

$y = -3x + \sqrt{2}$ (a *linear* function — it's a line with slope –3)

$y = 2x^2 - x + 9$ (a *quadratic* function — it's a parabola)

$f(x) = \sqrt[4]{2}\,x^3 - x^2$ (a *cubic* function)

$P(x) = -\pi x^4 + 5x^2 + 8$ (a *quartic* function)

$Q(x) = \frac{2}{3}x^5 + 1$ (a *quintic* function)

The key to any ***polynomial function*** is that all the exponents on the x come from $\mathbb{W}$, the set of whole numbers $\{0, 1, 2, 3, \ldots\}$. The coefficients (the numbers in front of the variables), on the other hand, can come from anywhere in $\mathbb{R}$, the set of real numbers.

Consider the quartic polynomial

$$y = -2\pi x^4 + \tfrac{9}{10}x^3 - 17x^2 + \sqrt{2}.$$

First, look at the exponents — they're all whole numbers. Even the last term of the polynomial, $\sqrt{2}$, can be written as $\sqrt{2}x^0$, and so even the exponent on this last term is a whole number. Thus, all the exponents (4, 3, 2, and 0) come from $\mathbb{W}$, while all the coefficients (-2π, $\frac{9}{10}$, -17, $\sqrt{2}$) come from $\mathbb{R}$. Considering the definition of polynomial, the given function is indeed a polynomial.

Each of the following is <u>not</u> a polynomial function:

$y = \dfrac{1}{x}$	($\frac{1}{x} = x^{-1}$ and $-1 \notin \mathbb{W}$)
$y = \sqrt{x}$	($\sqrt{x} = x^{1/2}$ and $\frac{1}{2} \notin \mathbb{W}$)
$f(x) = \dfrac{1}{\sqrt[3]{x}}$	($\frac{1}{\sqrt[3]{x}} = x^{-1/3}$ and $-\frac{1}{3} \notin \mathbb{W}$)
$g(x) = \lvert x-1 \rvert$	(no absolute values allowed around the x)
$E(x) = 2^x$	(since x is in the exponent, it can be any number)
$T(x) = \sin x$	(it's on your calculator, but it's not a polynomial)
$y = \log x$	(we'll learn this later -- it's not a polynomial)
$x^2 + y^2 = 25$	(it's a circle -- it's not a function of any kind)

Homework

1. Explain why $y = \pi x^5 - \sqrt{2}\,x^3 + \frac{1}{4}x - 17.5$ <u>is</u> a polynomial function.

2. Explain why $h(x) = \sqrt[3]{5}x^4 - \sqrt{2x} + \frac{1}{2}$ is <u>not</u> a polynomial function.

3. The highest exponent on the variable x in a polynomial is called the ***degree*** of the polynomial. Find the degree of each polynomial:

   a. $y = \pi$ b. $y = x^3 - 17x^2 + 8$

   c. $y = \sqrt{2}x^8 - \frac{\pi}{2}x^{10}$ d. $y = 7x + 5$

4. a. What is the domain of any polynomial function?

   b. T/F: Every parabola is a polynomial function.

   c. T/F: Every non-vertical line is a polynomial function.

   d. T/F: Every line is a polynomial function.

❑ *RATIONAL FUNCTIONS*

A ***rational function*** is simply the ratio of two polynomial functions. If P and Q are polynomial functions, then $R = \dfrac{P}{Q}$ is a rational function.

A typical example of a rational function is $y = \dfrac{3x^2 + 7x - 9}{x + 8}$.

<u>EXAMPLE 1:</u> **Graph:** $\boldsymbol{y = \dfrac{1}{x - 2}}$

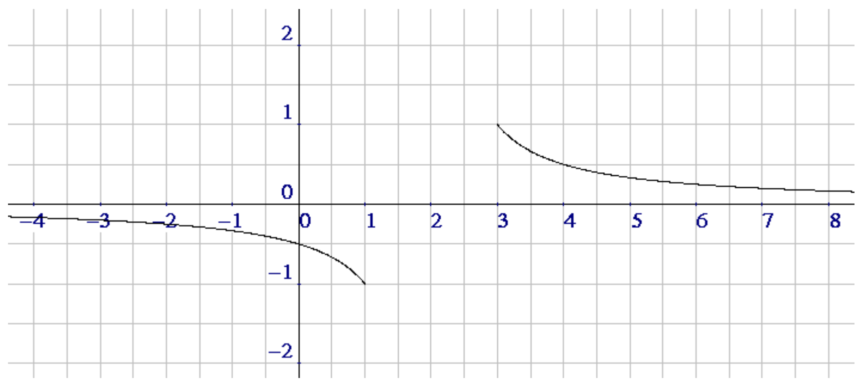
<u>Solution:</u> Since y is the ratio of a constant polynomial and a linear polynomial, we know that y is a rational function.

We begin our analysis of this rational function by determining the **domain**. In order that the fraction be defined, we must <u>not</u> divide by zero. What value of x makes the denominator zero? The value $x = 2$ will. (Just set $x - 2 = 0$ and solve for x.) Therefore, the domain is all real numbers except 2 -- that is, the domain is $\mathbb{R} - \{2\}$.

Intercepts come next. If $x = 0$, then $y = \frac{1}{0-2} = -\frac{1}{2}$. Thus, $(0, -\frac{1}{2})$ is the y-intercept. To find an x intercept, set $y = 0$. This gives $0 = \frac{1}{x-2} \Rightarrow 0(x-2) = \frac{1}{x-2}(x-2) \Rightarrow 0 = 1$, which has no solution. Thus, there are no x-intercepts.

Now for some ordered pairs that satisfy the formula $y = \frac{1}{x-2}$:
If we plot these points and connect them with a smooth curve, we will get the following graph:

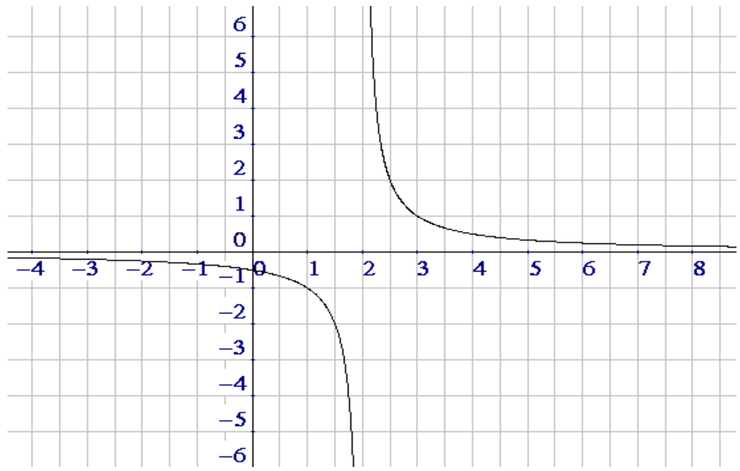
x	y
-3	$-\frac{1}{5}$
-2	$-\frac{1}{4}$
-1	$-\frac{1}{3}$
0	$-\frac{1}{2}$
1	-1
2	Und.
3	1
4	$\frac{1}{2}$
5	$\frac{1}{3}$
6	$\frac{1}{4}$
7	$\frac{1}{5}$



What some students do at this point is to simply connect the points (3, 1) and (1, −1) with a straight line. Talk about jumping to conclusions! Our domain of $\mathbb{R} - \{2\}$ implies that x cannot be 2 in this function; the straight-line trick won't work. So, we agree that a major chunk of the graph is missing. How do we get a better picture of the graph? We try some x-values that are near 2:

$$(1\tfrac{1}{2}, -2) \quad (1\tfrac{3}{4}, -4) \quad (1\tfrac{7}{8}, -8) \quad (2\tfrac{1}{2}, 2) \quad (2\tfrac{1}{4}, 4) \quad (2\tfrac{1}{8}, 8)$$

Adding these points to our previous attempt at a graph gives us a much better picture:



This graph has some really cool **limits**. Suppose we let x approach ∞. The y-values are positive (the curve is above the x-axis), but are getting smaller and smaller, approaching zero. Thus, as $x \to \infty$, $y \to 0$. Now let x approach $-\infty$. The y-values are negative but are rising toward zero. Therefore, as $x \to -\infty$, $y \to 0$.

The number 2 seems to be an interesting x-value. Although x can never be 2 in this function, it looks like the curve is getting closer and closer to the vertical line $x = 2$. In fact, if we let x approach 2 from the right, the curve grows taller and taller, and so we have the limit: As $x \to 2$ (from the right), $y \to \infty$. Now let x approach 2 from the left. This time, the curve drops rapidly toward negative infinity. This observation yields the limit: As $x \to 2$ (from the left), $y \to -\infty$. Let's summarize the four limits we've deduced:

As $x \to \infty$, $y \to 0$ As $x \to -\infty$, $y \to 0$

As $x \to 2$ (from the right), $y \to \infty$

As $x \to 2$ (from the left), $y \to -\infty$

From these four limits we can conclude that the **range** of the function is all real numbers except 0. That is, the range is $\mathbb{R} - \{0\}$.

Do you see that as you move far to the right or far to the left, the curve gets closer and closer to the x-axis? We say that the line $y = 0$ (which is the x-axis) is a ***horizontal asymptote***.

Now look at the region of the graph near $x = 2$. The curve gets closer and closer to the vertical line $x = 2$ (in fact, on both sides of the vertical line). We call the line $x = 2$ a ***vertical asymptote***.

EXAMPLE 2: **Graph:** $y = \dfrac{2x-1}{x+2}$

Solution: First we find the **domain**. Recall that this function will be undefined when the denominator is zero, which occurs when $x = -2$. Thus, the domain is $\mathbb{R} - \{-2\}$.

Now let's explore the **intercepts**:

If $x = 0$, then $y = \dfrac{2(0)-1}{0+2} = -\dfrac{1}{2}$. There's a y-intercept at $(0, -\frac{1}{2})$.

If $y = 0$, then $0 = \dfrac{2x-1}{x+2} \Rightarrow 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$. So $(\frac{1}{2}, 0)$ is an x-intercept.

It's time for some more ordered pairs for this function. Use your calculator to verify each of the following:

(–1, –3) (1, 0.33) (3, 1) (5, 1.29) (10, 1.58)

(15, 1.71) (20, 1.77) (100, 1.95) (1000, 1.995)

What's happening as x grows very large? It appears that y is approaching 2. That is, as $x \to \infty$, $y \to 2$.

Now we'll let x go the other direction:

(–3, 7) (–5, 3.67) (–10, 2.63) (–20, 2.28)

(–100, 2.05) (–1000, 2.01)

These points show that as $x \to -\infty$, $y \to 2$.

Finally, here are some ordered pairs for x's near –2 (the only real number not in the domain):

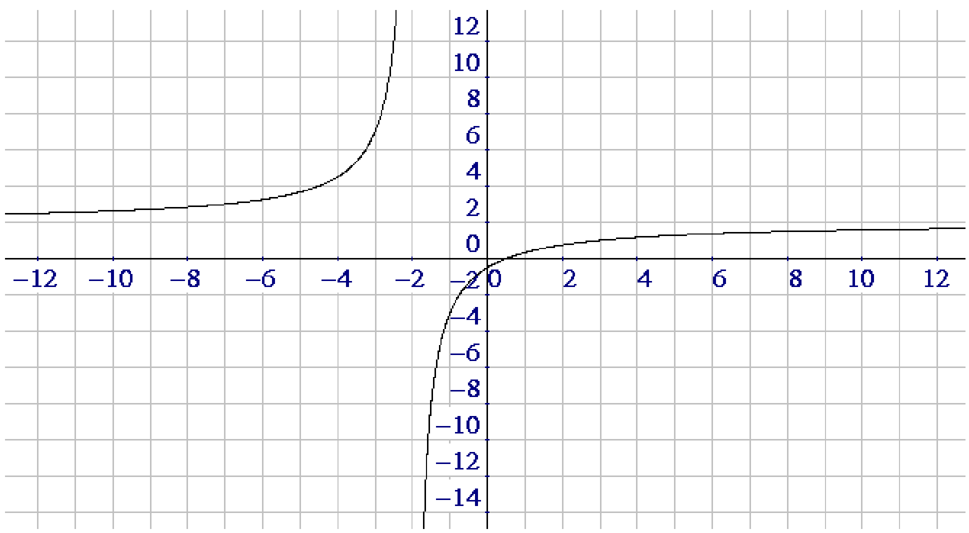
(–1.5, –8)   (–1.9, –48)   (–1.99, –498)

Thus, as $x \to -2$ (from the right), $y \to -\infty$.

(–2.5, 12)   (–2.1, 52)   (–2.01, 502)

Therefore, as $x \to -2$ (from the left), $y \to \infty$.

Plotting as many of the calculated points as possible, and taking into account the four limits we've found, the following graph emerges:



We can determine the **range** of the function by looking at the graph and seeing that y can be <u>any</u> value except 2. So the range is $\mathbb{R} - \{2\}$.

We can now be reasonably sure of the **asymptotes**. By recalling the limits described above or looking at the graph, we conclude that there's a vertical asymptote at $x = 2$ and a horizontal asymptote at $y = 2$.

EXAMPLE 3: **Graph:** $y = \dfrac{4}{1+x^2}$

Solution: Why is this function rational? Because it's the ratio P/Q of two polynomial functions: the constant polynomial $P(x) = 4$ and the quadratic polynomial $Q(x) = 1 + x^2$.

To find the **domain**, set the denominator to zero to see what's not in the domain: $1 + x^2 = 0$. This equation has no solution in $\mathbb{R}$, since solving it leads to $x = \pm\sqrt{-1}$, which are not real numbers. In fact, for any value of x, the quantity $1 + x^2$ is at least 1 (why?), so it certainly can't be zero. Since the denominator can never be zero, there's nothing to be excluded from the domain, and therefore the domain is $\mathbb{R}$. We can also figure that the graph will not have a **vertical asymptote**.

Let's check for y-axis **symmetry**: Replace x with $-x$:

$$y = \frac{4}{1+(-x)^2} = \frac{4}{1+x^2}$$

It's the same equation; our graph has y-axis symmetry. This means we need to use only x's that are ≥ 0 — that is, only points to the right of the y-axis need to be plotted. The graph on the left side of the y-axis will be a mirror image of the right side.

Now we seek the **intercepts**. Set $x = 0$ to get $y = 4$, and so the y-intercept is (0, 4). Now set $y = 0$, giving

$$0 = \frac{4}{1+x^2} \Rightarrow 0(1+x^2) = \frac{4}{1+x^2}(1+x^2) \Rightarrow 0 = 4.$$

This absurd result indicates that the equation has no solution; hence, there are no x-intercepts.

It's time for some ordered pairs for this function:

(1, 2) (2, 0.8) (3, 0.4) (4, 0.24) (10, 0.04) (200, 0.0001)

These points suggest the limit: As $x \to \infty$, $y \to 0$. This implies that $y = 0$ is a horizontal **asymptote**.

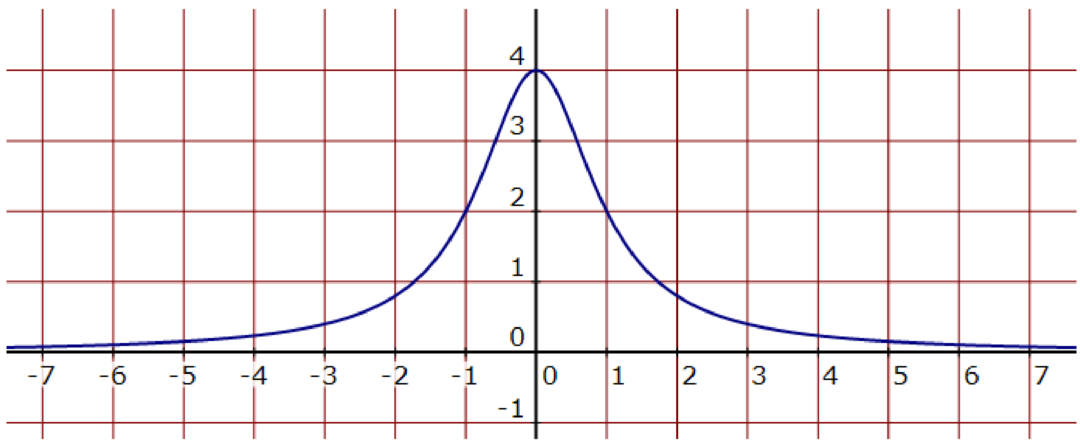
Here are some more ordered pairs, which allows us to see what happens as we approach the *y*-axis from the right:

(0.75, 2.56) (0.5, 3.2) (0.25, 3.76) (0.1, 3.96) (0.02, 3.998)

These points give us the limit:

As $x \to 0$ (from the right), $y \to 4$.

If we plot all the points calculated so far, and if we recall the *y*-axis symmetry, we get the following graph:



We determined at the outset that the domain of this rational function is $\mathbb{R}$. Is it clear from the graph that this is indeed the case?

The graph shows us that the **range** of the function is all real numbers between 0 and 4, excluding the 0 but including the 4. This set can be written $\{ y \in \mathbb{R} \mid 0 < y \le 4 \}$, or in interval notation as $(0, 4]$.

Homework

5. Consider the rational function in Example 2. Without referring to the graph, prove that y can have the value 2.01, but y can never have the value 2. Also without referring to the graph, verify that the graph does not have y-axis symmetry.

6. Find the domain:

   a. $y = \dfrac{2x+7}{9}$

   b. $f(x) = \dfrac{2x-3}{4+x}$

   c. $g(x) = \dfrac{3x}{2x-10}$

   d. $R(x) = \dfrac{x-1}{-7x+4}$

   e. $f(x) = \dfrac{x^2-9}{x^2-100}$

   f. $g(x) = \dfrac{8x-16}{x^2+25}$

7. Find the intercepts:

   a. $f(x) = \dfrac{x-4}{x-2}$

   b. $y = \dfrac{3}{5x-15}$

   c. $y = \dfrac{2x+1}{x-3}$

   d. $g(x) = \dfrac{5-x}{6x+1}$

8. Find the asymptotes:

   a. $R(x) = \dfrac{8x+1}{4x-4}$

   b. $y = \dfrac{2x-3}{2x+1}$

   c. $y = \dfrac{3x-7}{x+2}$

   d. $h(x) = \dfrac{2x+7}{4x-4}$

9. Find the domain, the intercepts, and the asymptotes of

$$y = \frac{1}{4+x^2}.$$

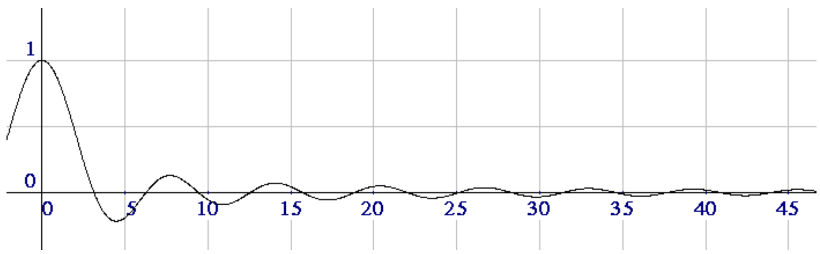
10. Perform a complete analysis of the function $y = \dfrac{2}{x-3}$.

11. Perform a complete analysis of the function $y = \dfrac{3x-5}{x-2}$.

12. Perform a complete analysis of the function $y = \dfrac{2}{2+x^2}$.

13. Perform a complete analysis of the function $y = \dfrac{-1}{x+1}$.

14. Consider the graph



Explain why the horizontal line $y = 0$ (the x-axis) is a horizontal asymptote for the curve.

Practice Problems

15. Explain why $f(x) = \sqrt{7}x^{10} + \pi x^7 - 6x - 1$ is a polynomial. What is its degree?

16. Explain why $y = 3x^5 - \sqrt{x} + \pi$ is not a polynomial.

17. a. A horizontal line (is, is not) a polynomial.
    b. The function $y = \dfrac{1}{x}$ (is, is not) a polynomial.
    c. What is the degree of the polynomial $y = 7x - \pi$?
    d. Is a circle a polynomial?

18. Consider the rational function $y = \dfrac{7}{2x-8}$.

a. Find the domain.
b. Find all the intercepts.
c. Find all the asymptotes.
d. Calculate y if $x = 4.1$.

19. Find all the intercepts and asymptotes of $r(x) = \dfrac{8x+6}{2x-3}$ and graph.

Notation Interruption:

Instead of writing

As $x \to 1$ (from the right), $y \to 5$,

we'll write

As $x \to 1^+$, $y \to 5$

Similarly, instead of

As $x \to 6$ (from the left), $y \to \infty$,

we'll say

As $x \to 6^-$, $y \to \infty$

20. Graph $y = \dfrac{-2x-2}{x-1}$. As $x \to 1^+$, $y \to$ ___. As $x \to 1^-$, $y \to$ ___.
As $x \to \infty$, $y \to$ ___. As $x \to -\infty$, $y \to$ ___.

21. Graph $y = \dfrac{5}{2+x^2}$. Discuss domain, range, symmetry, and asymptotes. As $x \to \infty$, $y \to$ ___. As $x \to -\infty$, $y \to$ ___.
As $x \to 0^+$, $y \to$ ___. As $x \to 0^-$, $y \to$ ___.

22. True/False:

a. $y = \sqrt[3]{7}x^{10} - \pi x^3 + \sqrt{2}$ is a polynomial.

b. $y = \frac{1}{x^5}$ is a polynomial.

c. The graph of $f(x) = \frac{1}{2x+10}$ has a vertical asymptote at $x = -5$.

d. The graph of $g(x) = \frac{10x+9}{5x-11}$ has a horizontal asymptote at $y = 10$.

e. The range of the function $y = \frac{6}{1+x^2}$ is (0, 6].

f. The above function has x-axis symmetry.

g. The above function has a domain of $\mathbb{R} - \{\pm 1\}$.

h. For the graph of $y = \frac{3x+1}{x-\pi}$, as $x \to \infty$, $y \to 3$.

Solutions

1. All coefficients are from $\mathbb{R}$, and all exponents are from $\mathbb{W}$.

2. The middle term is $\sqrt{2}\,x^{1/2}$, and $\frac{1}{2} \notin \mathbb{W}$.

3. a. 0 b. 3 c. 10 d. 1

4. a. $\mathbb{R}$ b. False c. True d. False

5. $y = \frac{2x-1}{x+2} \Rightarrow 2.01 = \frac{2x-1}{x+2} \Rightarrow 2.01x + 4.02 = 2x - 1 \Rightarrow x = -502$.
So, (–502, 2.01) is on the graph, and indeed y can be 2.01.

Now let's pretend that y could be 2; then
$2 = \frac{2x-1}{x+2} \Rightarrow 2x + 4 = 2x - 1 \Rightarrow 4 = -1 \Rightarrow$ no solution. Thus, there is no x which will make $y = 2$.

To test for y-axis symmetry, replace x with $-x$:

$y = \frac{2(-x)-1}{-x+2} \Rightarrow y = \frac{-2x-1}{-x+2}$, which is <u>not</u> the original formula, nor can it be made to be the original formula. You test for origin symmetry.

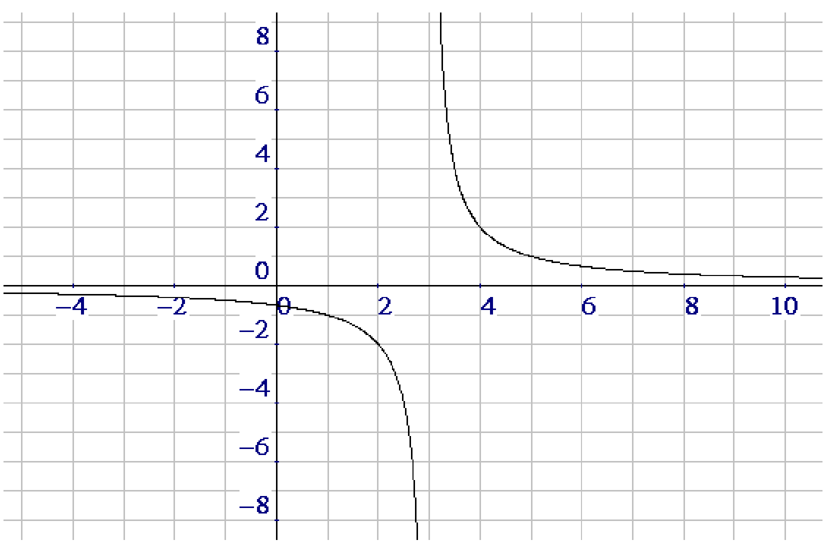
6. a. $\mathbb{R}$ b. $\mathbb{R} - \{-4\}$ c. $\mathbb{R} - \{5\}$ d. $\mathbb{R} - \left\{\frac{4}{7}\right\}$ e. $\mathbb{R} - \{\pm 10\}$ f. $\mathbb{R}$

7. a. (4, 0) (0, 2) b. $(0, -\frac{1}{5})$ c. $(-\frac{1}{2}, 0)$ $(0, -\frac{1}{3})$ d. (5, 0) (0, 5)

8. a. $x = 1$ $y = 2$ b. $x = -\frac{1}{2}$ $y = 1$ c. $x = -2$ $y = 3$ d. $x = 1$ $y = \frac{1}{2}$

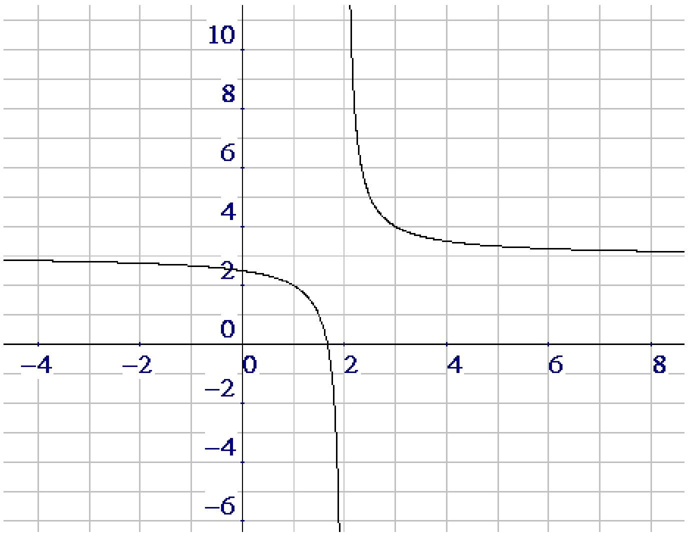
9. Since the only way the formula can be messed up is by dividing by 0, and since the denominator can never be zero (verify this yourself), the domain is $\mathbb{R}$.
Setting $x = 0$ gives a y-value of 1/4, so the y-intercept is $(0, \frac{1}{4})$. If you set $y = 0$, you'll get no solution for y. Thus, there is no x-intercept.
There are no vertical asymptotes, since the denominator is never zero.
Letting x approach either ∞ or $-\infty$, y approaches 0. Thus, a horizontal asymptote is $y = 0$ (the x-axis).

10.



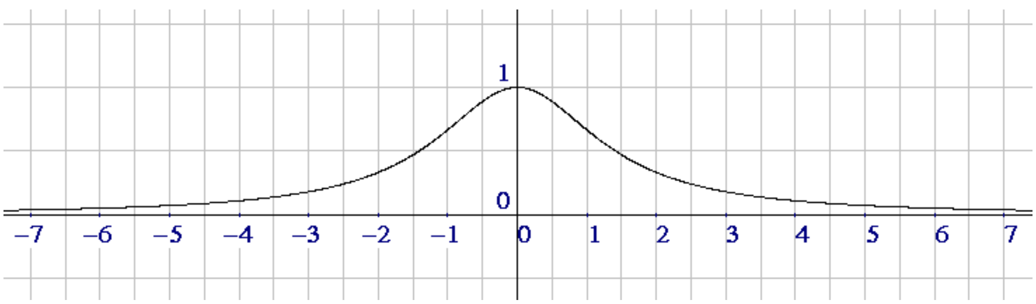
Domain $= \mathbb{R} - \{3\}$
x-int: none
y-int: $(0, -\frac{2}{3})$
As $x \to 3^+$, $y \to \infty$
As $x \to 3^-$, $y \to -\infty$
As $x \to \infty$, $y \to 0$
As $x \to -\infty$, $y \to 0$
vert asy: $x = 3$
horiz asy: $y = 0$
Range $= \mathbb{R} - \{0\}$
No symmetry

11.



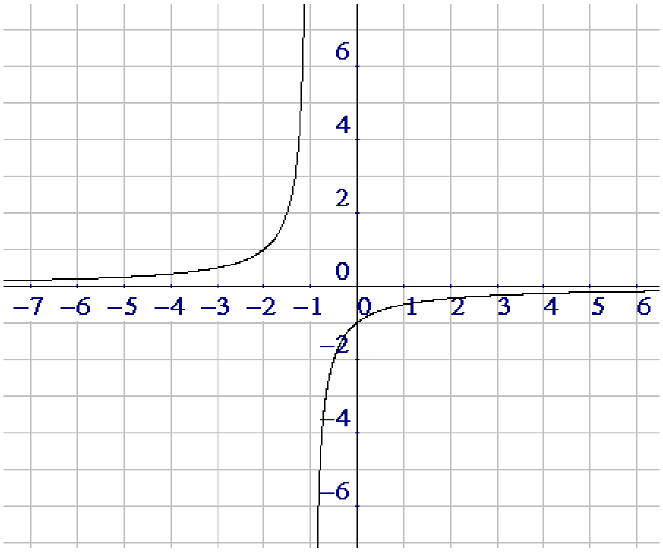
Domain = $\mathbb{R} - \{2\}$
x-int: $(\frac{5}{3}, 0)$
y-int: $(0, \frac{5}{2})$
As $x \to 2^+$, $y \to \infty$
As $x \to 2^-$, $y \to -\infty$
As $x \to \infty$, $y \to 3$
As $x \to -\infty$, $y \to 3$
vert asy: $x = 2$
horiz asy: $y = 3$
Range = $\mathbb{R} - \{3\}$
No symmetry

12.



Domain = $\mathbb{R}$ x-int: none y-int: (0, 1)   Symmetry: y-axis
As $x \to \infty$, $y \to 0$ As $x \to -\infty$, $y \to 0$
vert asy: none   horiz asy: $y = 0$
maximum point at (0, 1)
Range = $\{ y \in \mathbb{R} \mid 0 < y \le 1 \}$ = $(0, 1]$

13.



Domain = $\mathbb{R} - \{-1\}$
x-int: none
y-int: $(0, -1)$
As $x \to -1^{+}, y \to -\infty$
As $x \to -1^{-}, y \to \infty$
As $x \to \infty, y \to 0$
As $x \to -\infty, y \to 0$
vert asy: $x = -1$
horiz asy: $y = 0$
Range = $\mathbb{R} - \{0\}$
No symmetry

14. Because of the limit: As $x \to \infty, y \to 0$. Even though the graph intersects its own horizontal asymptote infinitely often, the curve nevertheless continues to get closer and closer to the x-axis (the line $y = 0$), and this is ultimately what is meant by a horizontal asymptote.

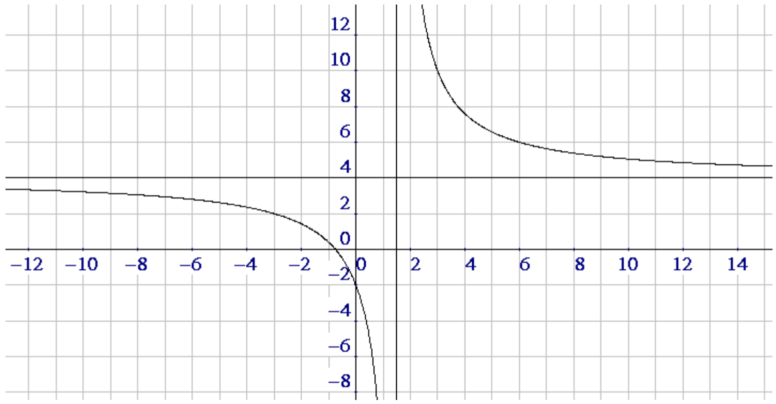
15. f is a polynomial because the coefficients are real numbers and the exponents (the 10, 7 and 1) are whole numbers. Its degree is 10.

16. Look at the middle term; it can written as $x^{1/2}$, a term whose exponent is <u>not</u> from the whole numbers.

17. a. is b. is not ($1/x = x^{-1}$) c. 1 d. It's not even a function, let alone the special function called a polynomial.

18. a. $\mathbb{R} - \{4\}$ b. $(0, -7/8)$ c. $x = 4$ and $y = 0$ d. 35

19. Intercepts: (0, –2) and $(-\frac{3}{4}, 0)$; vert asy: $x = \frac{3}{2}$; horiz asy: $y = 4$



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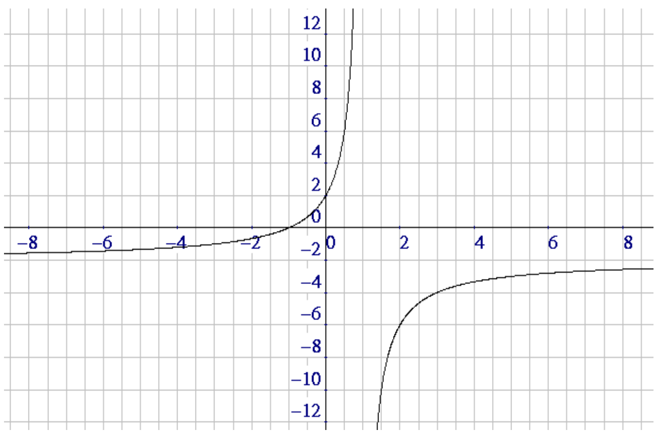
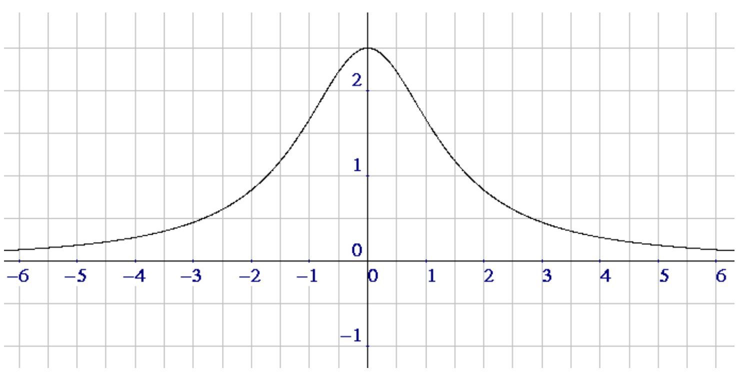


Limits: $-\infty$; ∞; -2; -2

21.



Domain = $\mathbb{R}$

Range = $(0, \frac{5}{2}]$

y-axis symmetry

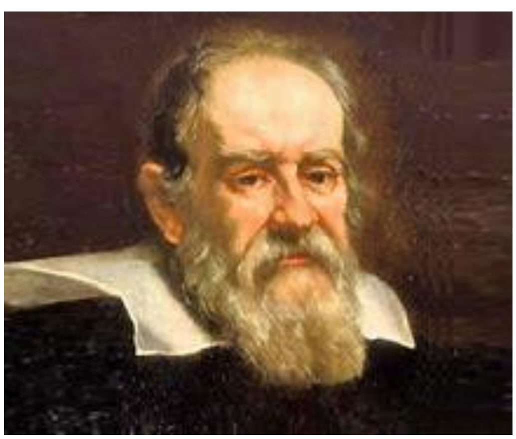
No vert asy

Horiz asy: $y = 0$

Limits: 0; 0; $\frac{5}{2}$; $\frac{5}{2}$

22. a. T b. F c. T d. F e. T f. F g. F h. T

" The universe cannot be read until we have learned the language and become familiar with the characters in which it is written. It is written in mathematical language."



Galileo Galiliei